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14. ABSTRACT

In this proposal, we considered certain invariant flows to treat a number of key issues in controlled active vision, in particular visual tracking. The flows are all derived from some variational principle and so are physically very well justified. In fact, many of the partial differential equations (PDE's) in imaging are based on curvature driven flows from interfacial physics. They have been shown to be useful for a number of applications including crystal growth, flame propagation, and computer vision. We have extensively studied the problems of optimal transport and optical flow for problems in tracking. Optimal transport has appeared in econometrics, fluid dynamics, automatic control, transportation, statistical physics, shape optimization, expert systems, and meteorology. In particular, for the general Visual tracking problem in controlled active vision, a robust and reliable object and shape recognition system is of major importance. We have based a new approach to this problem on the theory of optimal mass transport.

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Final Report for AFOSR Grant F49620-01-1-0017 Entitled "GEOMETRIC PDE'S AND INVARIANTS FOR PROBLEMS IN VISUAL CONTROL, TRACKING AND OPTIMIZATION"

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1 Introduction

In our just completed research program, we considered certain geometric variational methods for problems in controlled active vision. In this report, we will concentrate on the work we performed for the problems of registration, warping, and optical flow as related to dynamical tracking.

Vision is a key sensor modality in both the natural and man-made domains. The prevalence of biological vision in even very simple organisms, indicates its utility in man-made machines. More practically, cameras are in general rather simple, reliable passive sensing devices which are quite inexpensive per bit of data. Furthermore, vision can offer information at a high rate with high resolution with a wide field of view and accuracy capturing multi-spectral information. Finally cameras can be used in a more active manner. Namely, one can include motorized lenses mounted on mobile platforms which can actively explore the surroundings and suitably adapt their sensing capabilities.

For some time now, the role of control theory in vision has been recognized. In particular, the branches of control that deal with system uncertainty, namely adaptive and robust, have been proposed as essential tools in coming to grips with the problems of both biological and machine vision. These problems all become manifest when one attempts to use a visual sensor in an uncertain environment, and to feed back in some manner the information. These issues constitute a key thrust in our research program.

In this report, we will describe in some detail our new approach to the classical area of optimal transport. Optimal transport has appeared in econometrics, fluid dynamics, automatic control, transportation, statistical physics, shape optimization, expert systems,

and meteorology. In particular, for the general visual tracking problem in controlled active vision, a robust and reliable object and shape recognition system is of major importance. A key way to carry this out is via *template matching*, which is the matching of some object to another within a given catalogue of objects. Typically, the match will not be exact and hence some criterion is necessary to measure the "goodness of fit." The matching criterion can also be considered a *shape metric* for measuring the similarity between two objects. In our work, we applied concepts from optimal transport theory to develop such shape metrics.

This also led to a novel approach for the problem of optical flow. The computation of optical flow has proved to be an important tool for problems arising in active vision, including visual tracking. We have weakened the usual optical flow constraints with ideas from optimal transport and the theory of area-preserving mappings. This research represents our continuing efforts on the utilization visual information in a feedback loop.

2 Curvature Flows in Vision and Image Processing

The mathematical basis for our work rests on two pillars: invariant curvature driven flows and geometric variational problems. Here we will outline some of the basic results on the flows which are the basis of the partial differential equation methods in controlled active vision. We should note that these flows themselves are motivated by ideas from optimal control [42].

A curve may be regarded as a trajectory of a point moving in the plane. Formally, we define a (closed) curve $\mathcal{C}(\cdot)$ as the map $\mathcal{C}(p) : S^1 \to \mathbf{R}^2$ (where S^1 denotes the unit circle). We assume that our curves are have no self-intersections, i.e., are embedded.

We now consider plane curves deforming in time. Let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbb{R}^2$ denote a family of closed embedded curves, where t parameterizes the family, and p parameterizes each curve. Assume that this family evolves according to the following equation:

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \alpha \mathcal{T} + \beta \mathcal{N} \\ \mathcal{C}(p,0) = \mathcal{C}_0(p) \end{cases} \tag{1}$$

where $\mathcal{N}$ is the inward Euclidean unit normal, $\mathcal{T}$ is the unit tangent, and α and β are the tangent and normal components of the evolution velocity $\vec{\nu}$, respectively. In fact, it is easy to show that $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha \mathcal{T} + \beta \mathcal{N} \text{ and } \hat{\mathcal{C}}_t = \tilde{\beta} \mathcal{N},$$

respectively. (Here $\mathrm{Img}[\cdot]$ denotes the image of the given parameterized curve in $\mathbb{R}^2$.) Thus the tangential component affects only the parametrization, and not $\mathrm{Img}[\cdot]$ (which is independent of the parametrization by definition). Therefore, assuming that the normal component β of $\vec{\nu}$ (the curve evolution velocity) in (1) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta \mathcal{N}, \tag{2}$$

where $\beta = \vec{v} \cdot \mathcal{N}$.

The evolution (2) has been studied extensively for different functions β both from the theoretical and applied points of view. In particular, it was introduced into computer vision for a theory of shape in [39, 40, 41].

Two of the most important flows are derived for $\beta = \kappa$ and $\beta = 1$. In the former case, we will see that there is natural stochastic interpretation which may lead to an alternative way of evolving hypersurfaces, which is one of the subjects of our new research program.

More precisely, consider the flow

$$\frac{\partial \mathcal{C}}{\partial t} = \kappa \mathcal{N}. \tag{3}$$

Equation (3) has its origins in physical phenomena [23]. It is called the *geometric heat equation* or the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (3); see [23].

The second case is $\beta = 1$:

$$\frac{\partial \mathcal{C}}{\partial t} = \mathcal{N}. \tag{4}$$

This equation simulates, under certain conditions, the grassfire flow [9], and is the basis of the morphological scale-space defined by the disk as structuring element.

There is also an affine analogue of the Euclidean shortening flow which motivated the whole subject of invariant flows. Indeed, in [60], we show that the simplest non-trivial affine invariant flow in the plane is given by

$$\mathcal{C}_t = \kappa^{1/3} \mathcal{N}. \tag{5}$$

One can show that if $\mathcal{C}(\cdot, 0) : S^1 \to \mathbb{R}^2$ be a smooth embedded curve in the plane, then there exists a family $\mathcal{C} : S^1 \times [0, T) \to \mathbb{R}^2$ satisfying

$$\mathcal{C}_t = \kappa^{1/3} \mathcal{N},$$

such that $\mathcal{C}(\cdot, t)$ is smooth for all $t < T$, and moreover there is a $t_0 < T$ such that for all $t > t_0$, $\mathcal{C}(\cdot, t)$ is smooth and convex. Hence just as in the Euclidean case, a non-convex curve first becomes convex when evolving according to (5). After this, the curve converges to an ellipse from our results in [60].

One can use this flow to construct an *affine invariant scale-space* for planar shapes [61]. This in conjunction with the theory of differential invariants can be used for a theory of invariant object recognition.

3 Conformal (Geodesic) Active Contours

In this section, we will briefly review a paradigm for *snakes* or *active contours* based on principles from curvature driven flows and the calculus of variations. Active contours

may be regarded as autonomous processes which employ image coherence in order to track various features of interest over time. Such deformable contours have the ability to conform to various object shapes and motions. Snakes have been utilized for segmentation, edge detection, shape modelling, and visual tracking.

In the classical theory of snakes, one considers energy minimization methods where controlled continuity splines are allowed to move under the influence of external image dependent forces, internal forces, and certain constraints set by the user. As is well-known there may be a number of problems associated with this approach such as initializations, existence of multiple minima, and the selection of the elasticity parameters. Moreover, natural criteria for the splitting and merging of contours (or for the treatment of multiple contours) are not readily available in this framework. In [35], we propose a deformable contour model to successfully solve such problems which we call *conformal active contours.* (A similar approach was independently formulated in [13, 64]. In [13], the method is called *geodesic active contours.*)

The method is based on the Euclidean curve shortening evolution which defines the gradient direction in which a given curve is shrinking as fast as possible relative to Euclidean arc-length, and on the theory of conformal metrics. We multiply the Euclidean arc-length by a conformal factor defined by the features of interest which we want to extract, and then we compute the corresponding gradient evolution equations. The features which we want to capture therefore lie at the bottom of a potential well to which the initial contour will flow. Moreover, our model may be be extended to extract 3D contours based on motion by mean curvature [35, 45].

Briefly, let $g(x, y)$ be an image dependent function tailored to the type of feature which we want to capture. For example, the term $g(x, y)$ may chosen to be small near an edge, and so acts to stop the evolution when the contour gets close to an edge. The idea now is to change the ordinary arc-length function along a curve $\mathcal{C} = (x(p), y(p))^T$ with parameter p given by

$$ds = (x_p^2 + y_p^2)^{1/2} dp,$$

to

$$ds_g = (x_p^2 + y_p^2)^{1/2} g dp.$$

Then we want to compute the corresponding gradient flow for shortening length relative to the new metric ds_g.

Accordingly set

$$L_g(t) := \int_0^1 \| \frac{\partial \mathcal{C}}{\partial p} \| g dp.$$

Then taking the first variation of the modified length function L_g, and using integration by parts (see [35]), we get that

$$L'_g(t) \quad = \quad -\int_0^{L_g(t)} \langle \frac{\partial \mathcal{C}}{\partial t}, g\kappa\mathcal{N} - (\nabla g \cdot \mathcal{N})\mathcal{N} \rangle ds$$

which means that the direction in which the L_g perimeter is shrinking as fast as possible is given by

$$\frac{\partial \mathcal{C}}{\partial t} = (g\kappa - (\nabla g \cdot \mathcal{N}))\mathcal{N}. \tag{6}$$

This is precisely the gradient flow corresponding to the minimization of the length functional L_ϕ. The level set version of this is

$$\frac{\partial \Psi}{\partial t} = g\|\nabla\Psi\|\text{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nabla g \cdot \nabla\Psi. \tag{7}$$

One expects that this evolution should attract the contour very quickly to the feature which lies at the bottom of the potential well described by the gradient flow (7). We may also add a constant inflation term (see also [45] for a more rigorous justification), and so derive a modified model of (7) given by

$$\frac{\partial \Psi}{\partial t} = g\|\nabla\Psi\|(\text{div}(\frac{\nabla\Psi}{\|\nabla\Psi\|}) + \nu) + \nabla g \cdot \nabla\Psi. \tag{8}$$

4 Optimal Transport for Registration and Optical Flow

The mass transport problem was first formulated by Gaspar Monge in 1781, and concerned finding the optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one site to another. This problem was given a modern formulation in the work of Kantorovich [37], and so is now known as the *Monge–Kantorovich problem*. The problem of optimal tranpsort has appeared in econometrics, fluid dynamics, automatic control, transportation, statistical physics, shape optimization, expert systems, and meteorology [54]. It also naturally fits into certain problems in computer vision [25]. In particular, for the general visual tracking problem, a robust and reliable object and shape recognition system is of major importance. A key way to carry this out is via *template matching*, which is the matching of some object to another within a given catalogue of objects. Typically, the match will not be exact and hence some criterion is necessary to measure the "goodness of fit." For a description of various matching procedures, see [31] and the references therein. The matching criterion can also be considered a *shape metric* for measuring the similarity between two objects.

Registration proceeds in several steps. First, each image or data set to be matched should be individually calibrated, corrected for imaging distortions and artifacts, and cleared of noise. Next, a measure of similarity between the data sets must be established, so that one can quantify how close an image is from another after transformations are applied. Such a measure may include the similarity between pixel intensity values, as well as the proximity of predefined image features such as implanted fiducials, anatomical landmarks, surface contours, and ridge lines. Next, the transformation that maximizes the similarity between the transformed images is found. Often this transformation is given as

the solution of an optimization problem where the transformations to be considered are constrained to be of a predetermined class. Finally, once an optimal transformation is obtained, it is used to fuse the image data sets.

As we will explicate below, the method we propose for registration in the context of tracking is based on an optimization problem built around the L^2 Kantorovich–Wasserstein distance taken as the similarity measure. The constraint that we have put on the transformations considered is that they obey a mass preservation property. Thus, we match *mass densities* in this method, which may be thought of as weighted areas in 2D or weighted volumes in 3D.

We will also describe the use of such ideas for the problem of optical flow. The computation of optical flow has proved to be an important tool for problems arising in active vision, including visual tracking. The optical flow field is defined as the velocity vector field of apparent motion of brightness patterns in a sequence of images. We have explored various constrained optimization approaches for the purpose of accurately computing optical flow. In our work, we formulated approaches based upon the theory of optimal mass transport and area-preserving mappings.

4.1 Formulation of the Problem

We now give a modern formulation of the Monge–Kantorovich problem. Let Ω_0 and Ω_1 be two subdomains of $\mathbb{R}^d$, with smooth boundaries, each with a positive density function, μ_0 and μ_1, respectively. We assume

$$\int_{\Omega_0} \mu_0 = \int_{\Omega_1} \mu_1$$

so that the same total mass is associated with Ω_0 and Ω_1. We consider diffeomorphisms $\tilde{u}$ from (Ω_0, μ_0) to (Ω_1, μ_1) which map one density to the other in the sense that

$$\mu_0 = |D\tilde{u}|\, \mu_1 \circ \tilde{u}, \tag{9}$$

which we will call the *mass preservation* (MP) property, and write $\tilde{u} \in MP$. Equation (9) is called the *Jacobian equation.* Here $|D\tilde{u}|$ denotes the determinant of the Jacobian map $D\tilde{u}$. In particular, Equation (9) implies, for example, that if a small region in Ω_0 is mapped to a larger region in Ω_1, then there must be a corresponding decrease in density in order for the mass to be preserved. A mapping $\tilde{u}$ that satisfies this property may thus be thought of as defining a redistribution of a mass of material from one distribution μ_0 to another distribution μ_1.

There may be many such mappings, and we want to pick out an optimal one in some sense. Accordingly, we define the L^p Kantorovich–Wasserstein metric as follows:

$$d_p(\mu_0, \mu_1)^p := \inf_{\tilde{u} \in MP} \int \|\tilde{u}(x) - x\|^p \mu_0(x)\, dx. \tag{10}$$

An *optimal MP map*, when it exists, is one which minimizes this integral. This functional is seen to place a penalty on the distance the map $\tilde{u}$ moves each bit of material, weighted by the material's mass.

The case $p = 2$ has been extensively studied. The L^2 Monge–Kantorovich problem has been studied in statistics, functional analysis, and the atmospheric sciences; see [17, 8] and the references therein. A fundamental theoretical result [43, 10, 27], is that there is a unique optimal $\tilde{u} \in MP$ transporting μ_0 to μ_1, and that this $\tilde{u}$ is characterized as the gradient of a convex function w, i.e., $\tilde{u} = \nabla w$. Note that from Equation (9), we have that w satisfies the *Monge–Ampère* equation

$$|Hw|\, \mu_1 \circ (\nabla w) = \mu_0,$$

where $|Hw|$ denotes the determinant of the Hessian Hw of w.

Hence, the Kantorovich–Wasserstein metric defines the distance between two mass densities, by computing the cheapest way to transport the mass from one domain to the other with respect to the functional given in (10), the optimal transport map in the $p = 2$ case being the gradient of a certain function. The novelty of this result is that like the Riemann mapping theorem in the plane, the procedure singles out a particular map with preferred geometry.

4.2 Monge-Kantorovich and Optimal Transport

In our research, we focused on the uses of ideas from optimal transport for problems in controlled active vision and visual tracking. However, given the potential power of these ideas in systems and control, we would like to list some key uses of Monge-Kantorovich:

1. Lyapunov theory is essential is studying nonlinear system stability and controller synthesis. In some very interesting work, Rantzer [55] has formulated a dual to Lyapunov's second theorem. The idea is that the Lyapunov function is regarded as the "cost to go" in an optimal transport problem and is dual (in the sense of linear programming) to the density function typically studied in Monge-Kantorovich theory. These ideas give a powerful new tool in studying nonlinear system analysis.

2. Shape optimization is another area of use for optimal transport. For example, given two densities and an insulating medium into which we place a fixed amount of conducting material one can consider the problem of the optimal placement of the conducting material to minimize the heating induced by the flow. This can be put into the Monge-Kantorovich optimal transport framework. Similar remarks apply to problems in compression molding, where one considers an incompressible plastic material being pressed being two plates in which one wants to track the air-plastic interface.

3. One of the most beautiful uses of optimal transport is in meteorology, in particular semigeostrophic models. These are concerned with with large scale stratified flows with front formation [17]. The idea is that meteorologists want to model how fronts

arise in large-scale weather patterns. Tracking such fronts is a key goal, and semi-geostrophic equations seem to give a reasonable mathematical model for the creation of such fronts. This leads naturally to optimal mass transport equations.

4.3 Background on Algorithms for Computing The Transport Map

There have been a number of algorithms considered for computing an optimal transport map. For example, methods have been proposed based on linear programming [54], and on Lagrangian mechanics closely related to ideas from the study of fluid dynamics [8]. An interesting geometric method has been formulated by Cullen and Purser [17].

One very common method is to reduce the L^2 optimal transport to a linear programming problem. Thus one can approximate the the densities μ_0 and μ_0 as weighted sums delta functions, such as

$$\mu_0(x) = 1/N \sum_{i=1,N} \delta(x - x_i), \quad \mu_1(x) = 1/N \sum_{i=1,N} \delta(x - y_i),$$

for $2N$ give points $x_1, \ldots, x_N, y_1, \ldots, y_N \in \mathbf{R}^d$. The L^2 Kantorovich distance is then

$$d_2(\mu_0, \mu_1)^2 = \inf_\sigma \sum_1^N |y_i - x_{\sigma(i)}|^2, \tag{11}$$

where the infimum is over all permutations on N letters σ.

This problem can be solved as linear programming problem, by noting that Equation (11) can be expressed as

$$\inf_\rho \sum_{i,j=1}^N c_{ij}\rho_{ij}$$

where $c_{ij} = |x_i - y_j|^2$, and ρ denotes any $N \times N$ matrix with non-negative entries, such that the sum of all columns and rows equals 1 (these are the "doubly-stochastic matrices"). There are optimal algorithms for general cost matrices c_{ij}, but to the best of our knowledge there are no known optimal algorithms for the special case in which $c_{ij} = |x_i - y_j|^2$. Finally, note that even in the 2D case, typical image sizes run to 512×512, and so the linear programming problem can get to be quite unwieldy.

A more effective algorithm based on ideas from continuum mechanics was proposed in [8]. This based on ideas from Lagrangian mechanics and a certain relaxation method. This has influenced our approach discussed below. In our case however for image tracking, we will argue that the most effective method should be based on gradient descent and the concept of "polar factorization." We discuss this in the next section.

4.4 Variational Algorithms for Optimal Transport

In this section, we describe a natural solution to L^2 Monge-Kantorovich based on the equivalent problem of *polar factorization*; see [10, 26, 48] and the references therein. The mathematical basis for the approach described here may be found in [5], and applications to visual tracking in [30]. We will work with the general case of subdomains in $\mathbb{R}^d$, and point out some simplifications that are possible for the $\mathbb{R}^2$ case.

As above, let Ω_0, $\Omega_1 \subset \mathbb{R}^d$ be subdomains with smooth boundaries, with corresponding positive density functions μ_0 and μ_1 satisfying $\int_{\Omega_0} \mu_0 = \int_{\Omega_1} \mu_1$. Let $u : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$ be an initial mapping with the mass preserving (MP) property. Then according to the generalized results of [10, 26], one can write

$$u = (\nabla w) \circ s, \tag{12}$$

where w is a convex function and s is an MP mapping $s : (\Omega_0, \mu_0) \to (\Omega_0, \mu_0)$. This is the *polar factorization* of u with respect to μ_0. In [26], just the case of area preservation is considered, i.e., μ_0 is assumed constant, but the general case goes through as well.

Our goal is to find the polar factorization of the MP mapping u, according to the following strategy. We consider the family of MP mappings of the form $\tilde{u} = u \circ s^{-1}$ as s varies over MP mappings from (Ω_0, μ_0) to itself. If we consider $\tilde{u}$ as a vector field, we can always find a function w and another vector field χ, with $\mathrm{div}(\chi) = 0$, such that

$$\tilde{u} = \nabla w + \chi,$$

i.e., we can decompose $\tilde{u}$ into the sum of a curl-free and divergence-free vector field [66]. Thus, what we try to do is find a mapping s which will yield a $\tilde{u}$ without any curl, that is, such that $\tilde{u} = \nabla w$. Once such an s is found, we will have $u = \tilde{u} \circ s = (\nabla w) \circ s$ and so we will have found the polar factorization (12) of our given function u.

Now, here is the key point. As we discussed above, the unique optimal solution of the L^2 Monge–Kantorovich problem has the form $\tilde{u} = \nabla w$, and so the problem of finding the polar factorization of u and finding the optimal Monge–Kantorovich mapping $\tilde{u}$ are equivalent. In essence, our proposed approach to solve the Monge–Kantorovich problem is to create a "rearrangement" of an initial vector field u using a map s, so that the resulting vector field $\tilde{u} = u \circ s^{-1}$ has no curl.

Finding an Initial Mapping:

We will describe an explicit algorithm to solve the Monge–Kantorovich problem. So we want to minimize the L^2 Kantorovich–Wasserstein distance functional over MP functions from (Ω_0, μ_0) to (Ω_1, μ_1). We will try to do this by finding an initial MP mapping u and then minimizing over $\tilde{u} = u \circ s^{-1}$ by varying s over MP mappings from Ω_0 to Ω_0, starting with s equal to the identity map. Our first task is to find and initial MP mapping u. This can be done for general domains using a method of Moser [51, 18], or for simpler domains using the following algorithm. For simplicity, we work in $\mathbb{R}^2$ and assume $\Omega_0 = \Omega_1 = [0, 1]^2$, the generalization to higher dimensions being straightforward. We define a function $a = a(x)$

by the equation

$$\int_0^{a(x)} \int_0^1 \mu_1(\eta, y)\ dy\ d\eta = \int_0^x \int_0^1 \mu_0(\eta, y)\ dy\ d\eta \tag{13}$$

which gives by differentiation with respect to x

$$a'(x) \int_0^1 \mu_1(a(x), y)\ dy = \int_0^1 \mu_0(x, y)\ dy. \tag{14}$$

We may now define a function $b = b(x, y)$ by the equation

$$a'(x) \int_0^{b(x,y)} \mu_1(a(x), \rho)\ d\rho = \int_0^y \mu_0(x, \rho)\ d\rho, \tag{15}$$

and set $u(x, y) = (a(x), b(x, y))$. Since $a_y = 0$, $|Du| = a_x b_y$, and differentiating (15) with respect to y we find

$$\begin{aligned} a'(x)\ b_y(x, y)\ \mu_1(a(x), b(x, y)) &= \mu_0(x, y) \\ |Du|\ \mu_1 \circ u &= \mu_0, \end{aligned}$$

which is the MP property we need. This process can be interpreted as the solution of a one-dimensional Monge–Kantorovich problem in the x direction followed by the solution of a family of one-dimensional Monge–Kantorovich problems in the y direction.

The map created above can be quite crude, and so one can use other methods for writing down an initial mapping including using an approach based on volume-preserving diffeomorphisms as suggested in [51].

Removing the Curl:

Once an initial MP u is found, we need to apply the process which will remove its curl. We first note that the composition of two mass preserving (MP) mappings is an MP mapping, and the inverse of an MP mapping is an MP mapping. Thus, since u is an MP mapping, we have that $\tilde{u} = u \circ s^{-1}$ is an MP mapping if and only if s is, that is, if and only if

$$\mu_0 = |Ds|\, \mu_0 \circ s.$$

In particular, when μ_0 is constant, this equation requires that s be area or volume preserving.

Next, rather than working with s directly, we solve the polar factorization problem via gradient descent. Accordingly, we will assume that s is a function of time, and then determine what s_t should be to decrease the L^2 Monge–Kantorovich functional. This will give us an evolution equation for s and in turn an equation for $\tilde{u}_t$ as well, the latter being the most important for implementation. By differentiating $\tilde{u} \circ s = u$ with respect to time, we get

$$\tilde{u}_t = -D\tilde{u}\, \tilde{s}_t$$

where we've abused notation to define $\tilde{s}_t := s_t \circ s^{-1}$. We need to make sure that s maintains its MP property. Differentiating $\mu_0 = |Ds|\, \mu_0 \circ s$ with respect to time, we derive

$$\operatorname{div}(\mu_0\, \tilde{s}_t) = 0,$$

from which we see that $\tilde{s}_t$, s_t and $\tilde{u}_t$ should have the following forms:

$$\tilde{s}_t = \frac{1}{\mu_0}\zeta, \tag{16}$$

$$s_t = \left(\frac{1}{\mu_0}\zeta\right) \circ s, \tag{17}$$

$$\tilde{u}_t = -\frac{1}{\mu_0} D\tilde{u}\, \zeta, \tag{18}$$

for some vector field ζ on Ω_0, with $\operatorname{div}(\zeta) = 0$ and $\langle \zeta, \vec{n} \rangle = 0$ on $\partial\Omega_0$, $\vec{n}$ being the normal to the boundary of Ω_0. This last condition ensures that s remains a mapping from Ω_0 to itself, by preventing the flow of s, given by $s_t = \left(\frac{1}{\mu_0}\zeta\right) \circ s$, from crossing the boundary of Ω_0. This also means that the range of $\tilde{u} = u \circ s^{-1}$ is always $u(\Omega_0) = \Omega_1$.

Consider now the problem of minimizing the Monge–Kantorovich functional:

$$M = \int ||\tilde{u} - x||^2 \mu_0 \tag{19}$$

$$= \int ||\tilde{u}||^2 \mu_0 - 2\int \langle \tilde{u}, x \rangle\, \mu_0 + \int ||x||^2 \mu_0. \tag{20}$$

The last term is obviously independent of time. Interestingly, so is the first,

$$\begin{aligned} \int ||\tilde{u}||^2 \mu_0 &= \int ||u \circ s^{-1}||^2 \mu_0 \\ &= \int ||u \circ s^{-1}||^2\, |Ds^{-1}|\, \mu_0 \circ s^{-1} \\ &= \int ||u||^2 \mu_0 \end{aligned}$$

where $\mu_0 = |Ds^{-1}|\, \mu_0 \circ s^{-1}$ since s^{-1} is an MP map.

Turning now to the middle term, we do a similar trick,

$$\begin{aligned} \int \langle \tilde{u}, x \rangle\, \mu_0 &= \int \left\langle u \circ s^{-1}, s \circ s^{-1} \right\rangle \mu_0 \\ &= \int \left\langle u \circ s^{-1}, s \circ s^{-1} \right\rangle |Ds^{-1}|\, \mu_0 \circ s^{-1} \\ &= \int \langle u, s \rangle\, \mu_0, \end{aligned}$$

and taking $s_t = \left(\frac{1}{\mu_0}\zeta\right) \circ s$, we compute

$$
\begin{aligned}
-\frac{1}{2} M_t &= \int \langle u, s_t \rangle \; \mu_0 \\
&= \int \left\langle \tilde{u} \circ s, \left(\frac{1}{\mu_0}\zeta\right) \circ s \right\rangle \mu_0 \\
&= \int \left\langle \tilde{u} \circ s, \left(\frac{1}{\mu_0}\zeta\right) \circ s \right\rangle |Ds| \, \mu_0 \circ s \\
&= \int \left\langle \tilde{u}, \frac{1}{\mu_0}\zeta \right\rangle \mu_0 \\
&= \int \langle \tilde{u}, \zeta \rangle .
\end{aligned}
$$

Now decomposing $\tilde{u}$ as $\tilde{u} = \nabla w + \chi$, we have

$$
\begin{aligned}
-\frac{1}{2} M_t &= \int \langle \nabla w + \chi, \zeta \rangle \\
&= \int \langle \nabla w, \zeta \rangle + \int \langle \chi, \zeta \rangle \\
&= \int (\mathrm{div}(w\zeta) - w \, \mathrm{div}(\zeta)) + \int \langle \chi, \zeta \rangle \\
&= \int_{\partial\Omega_0} w \; \langle \zeta, n \rangle + \int \langle \chi, \zeta \rangle \\
&= \int \langle \chi, \zeta \rangle ,
\end{aligned}
$$

where we've used the divergence theorem, $\mathrm{div}(\zeta) = 0$, and $\langle \zeta, \vec{n} \rangle = 0$ on $\partial\Omega_0$. Thus, in order to decrease M, we can take $\zeta = \chi$ with corresponding formulas (16)-(18) for s_t, $\tilde{s}_t$, and $\tilde{u}_t$, provided that we have $\mathrm{div}(\chi) = 0$ and $\langle \chi, \vec{n} \rangle = 0$ on $\partial\Omega_0$. Thus it remains to show that we can decompose $\tilde{u}$ as $\tilde{u} = \nabla w + \chi$ for such a χ.

Gradient Descent: $\mathbb{R}^d$:

We let w be a solution of the Neumann-type boundary problem

$$
\begin{aligned}
\mathrm{div}(\tilde{u}) &= \Delta w & (21) \\
\langle \nabla w, \vec{n} \rangle &= \langle \tilde{u}, \vec{n} \rangle \text{ on } \partial\Omega_0, & (22)
\end{aligned}
$$

and set $\chi = \tilde{u} - \nabla w$. It is then easily seen that χ satisfies the necessary requirements.

Thus, by (18), we have the following evolution equation for $\tilde{u}$:

$$
\tilde{u}_t = -\frac{1}{\mu_0} D\tilde{u} \left(\tilde{u} - \nabla \Delta^{-1} \mathrm{div}(\tilde{u}) \right) . \tag{23}
$$

This is a first order **non-local** scheme for $\tilde{u}_t$ if we count Δ^{-1} as minus 2 derivatives. Note that this flow is consistent with respect to the Monge–Kantorovich theory in the following sense. If $\tilde{u}$ is optimal, then it is given as $\tilde{u} = \nabla w$, in which case $\tilde{u} - \nabla\Delta^{-1}\mathrm{div}(\tilde{u}) = \nabla w - \nabla\Delta^{-1}\mathrm{div}(\nabla w) = 0$ so that by (23), $\tilde{u}_t = 0$.

Gradient Descent: $\mathbb{R}^2$:

The situation is somewhat simpler in the $\mathbb{R}^2$ case, due to the fact that a divergence free vector field χ can in general be written as $\chi = \nabla^{\perp} h$ for some scalar function h, where $\perp$ represents rotation by 90 deg, so that $\nabla^{\perp} h = (-h_y, h_x)$. In this case, (21) becomes

$$-\frac{1}{2}M_t = \int \langle \nabla^{\perp} f, \nabla^{\perp} h \rangle = \int \langle \nabla f, \nabla h \rangle \tag{24}$$

where the decomposition of $\tilde{u}$ is $\tilde{u} = \nabla w + \nabla^{\perp} f$, and we can take $h = f$. The function f can be found by solving the Dirichlet-type boundary problem

$$\begin{aligned} -\mathrm{div}(\tilde{u}^{\perp}) &= \Delta f, & (25)\\ f &= 0 \text{ on } \partial\Omega_0, & (26) \end{aligned}$$

which gives us the evolution equation

$$\tilde{u}_t = \frac{1}{\mu_0} D\tilde{u}\, \nabla^{\perp}\Delta^{-1}\mathrm{div}(\tilde{u}^{\perp}). \tag{27}$$

We may also derive a second order **local** evolution equation for $\tilde{u}$ by using the divergence theorem with (24) to get

$$\tilde{u}_t = -\frac{1}{\mu_0} D\tilde{u}\, \nabla^{\perp}\mathrm{div}(\tilde{u}^{\perp}), \tag{28}$$

where appropriate handling of the evolution at the boundary is necessary. If the boundary is square, then one natural thing to do would be to assume that the displacement map $\tilde{u} - x$ is periodic.

Defining the Warping Map:

Typically in elastic registration, one wants to see an explicit warping which smoothly deforms one image into the other. This can easily be done using the solution of the Monge–Kantorovich problem. Thus, we assume now that we have applied our gradient descent process as described above and that it has converged to the Monge–Kantorovich mapping $\tilde{u}_{MK}$.

Following the work of Benamou and Brenier, [8], (see also [27]), we consider the following related problem:

$$\inf \int \int_0^1 \mu(t,x) \|v(t,x)\|^2 \, dt\, dx$$

over all time varying densities μ and velocity fields v satisfying

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu v) = 0, \tag{29}$$

$$\mu(0,\cdot) = \mu_0, \quad \mu(1,\cdot) = \mu_1. \tag{30}$$

It is shown in [8] that this infimum is attained for some μ_{min} and v_{min}, and that it is equal to the L^2 Kantorovich–Wasserstein distance between μ_0 and μ_1. Further, the flow $X = X(x,t)$ corresponding to the minimizing velocity field v_{min} via

$$X(x,0) = x, \quad X_t = v_{min} \circ X$$

is given simply as

$$X(x,t) = x + t\ (\tilde{u}_{MK}(x) - x). \tag{31}$$

Note that when $t = 0$, X is the identity map and when $t = 1$, it is the solution $\tilde{u}_{MK}$ to the Monge–Kantorovich problem. This analysis provides appropriate justification for using (31) to *define* our continuous warping map X between the densities μ_0 and μ_1.

4.5 More General Cost Functions

We can also consider more general cost functions which are useful in image interpolation and optical flow; see Section 5 below. Indeed, the theory we described above extends formally quite easily to general cost functions [5]. Indeed, suppose we are trying to minimize a functional

$$M = \int \Phi(\tilde{u}(x) - x)\mu_0(x)\, dx \tag{32}$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a positive C^1 cost function.

From the above argument, we know that the evolution of $\tilde{u} : \mathbb{R}^d \to \mathbb{R}^d$ should be given as

$$\tilde{u}_t = -\frac{1}{\mu_0} D\tilde{u} \cdot \zeta, \tag{33}$$

where ζ is a divergence free vector field. Taking the first variation, one can derive an expression of the form

$$M_t = -\int \langle \Phi'(\tilde{u}(x) - x), \zeta \rangle . \tag{34}$$

Next decomposing

$$\Phi'(\tilde{u}(x) - x) = \nabla w + \chi, \tag{35}$$

we have

$$\begin{aligned} M_t &= -\int \langle \chi + \nabla w, \zeta \rangle \\ &= -\int \langle \chi, \zeta \rangle , \end{aligned}$$

by the divergence theorem. Thus in order to decrease M we can take $\zeta = \chi$, provided we show that $\operatorname{div}(\chi) = 0$ and $\langle \chi, \vec{n} \rangle = 0$. So we need to solve

$$\begin{aligned} \operatorname{div}(\Phi'(\tilde{u}(x) - x)) &= \nabla w, \\ \langle \nabla w, \vec{n} \rangle &= \langle \Phi'(\tilde{u} - x), \vec{n} \rangle \text{ on } \partial\Omega_0 \end{aligned}$$

for w. This leads to the following *nonlocal* equation:

$$\tilde{u}_t = -\frac{1}{\mu_0} D\tilde{u} \cdot (I - \nabla \Delta^{-1} \nabla \cdot) \Phi'(\tilde{u} - x).$$

A similar argument to that given above also will give a local gradient descent scheme in this case as well [5].

5 Area-Preserving Diffeomorphisms and Optical Flow

The computation of optical flow has proved to be an important tool for problems arising in active vision. The optical flow field is the velocity vector field of apparent motion of brightness patterns in a sequence of images [34]. One assumes that the motion of the brightness patterns is the result of relative motion, large enough to register a change in the spatial distribution of intensities on the images. Thus, relative motion between an object and a camera can give rise to optical flow. Similarly, relative motion among objects in a scene being imaged by a static camera can give rise to optical flow. The problem of computing optical flow is ill-posed and so well-posedness has to be imposed by assuming suitable *a priori* knowledge. For example, a number of researchers have considered a variational formulation for imposing such *a priori* knowledge.

One constraint which has often been used in the literature is the "optical flow constraint"(OFC). The OFC is a result of the simplifying assumption of constancy of the intensity, $I = I(x, y, t)$, at any point in the image [34]. It can be expressed as the following linear equation in the unknown variables u and v

$$I_x u + I_y v + I_t = 0, \tag{36}$$

where I_x, I_y and I_t are the intensity gradients in the x, y, and the temporal directions respectively, and u and v are the x and y velocity components of the apparent motion of brightness patterns in the images, respectively. It has been shown that the OFC holds provided the scene has Lambertian surfaces and is illuminated by either a uniform or an

isotropic light source, the 3-D motion is translational, the optical system is calibrated and the patterns in the scene are locally rigid.

It is not difficult to see from equation (36) that computation of optical flow is unique only up to computation of the flow along the intensity gradient $\nabla I = (I_x, I_y)^T$ at a point in the image [34]. (The superscript T denotes "transpose.") This is the celebrated *aperture problem.* One way of treating the aperture problem is through the use of regularization in computation of optical flow, and consequently the choice of an appropriate constraint. A natural choice for such a constraint is the imposition of some measure of consistency on the flow vectors situated close to one another on the image.

In their pioneering work, Horn and Schunk [34] use a quadratic smoothness constraint. The immediate difficulty with this method is that at the object boundaries, where it is natural to expect discontinuities in the flow, such a smoothness constraint will have difficulty capturing the optical flow. For instance, in the case of a quadratic constraint in the form of the square of the norm of the gradient of the optical flow field [34], the Euler-Lagrange (partial) differential equations for the velocity components turn out to be *linear* elliptic. The corresponding parabolic equations therefore have a linear diffusive nature, and tend to blur the edges of a given image. In the past, work has been done to try to suppress such a constraint in directions orthogonal to the occluding boundaries in an effort to capture discontinuities in image intensities that arise on the edges. In [44] a total variational type optimization problem is proposed in which the resulting Euler-Lagrange equations are nonlinear geometric heat equations which preserve edges much better.

The optical flow constraint above is of course very strong. Motivated by Moser [51], we have proposed a modification of this that also could be placed in a variational setting. Namely, the Moser construction described in [51] allows one to do the following: Given a family of nowhere-zero 2-forms τ_t, we have an explicit method to determine a family of diffeomorphisms ϕ_t such that

$$\phi_t^* \tau_t = \tau_0.$$

Differentiating $\phi_t^* \tau_t = \tau_0$ with respect to t yields

$$\frac{\partial}{\partial t}\tau_t + \langle \nabla \tau_t, u_t \rangle + \tau_t \mathrm{div}(u_t) = 0. \tag{37}$$

This is very similar in form to the standard optical flow constraint with the divergence term added. In our work, we interpret image intensity as a type of form, and apply the Moser analysis under a much less restrictive assumption than the standard optical flow constraint given in equation (36).

6 Image Interpolation and Optical Flow

We would like to show how optimal transport ideas can be used to interpolate a given scalar field (and even a vector or tensor field) from one image to another, as well as in optical flow. For simplicity, we just consider the intensity maps in our discussion in this section. We have already alluded to using area-preserving maps to weaken the optical

flow constraint. In the present approach, we use the optimal transport philosophy more directly.

The idea is to minimize a functional of the following form over area-preserving maps:

$$M = \int (F \circ \tilde{u} - G)^2 + \alpha^2 \int \|D\tilde{u}\|^2. \tag{38}$$

Here the first term controls the "goodness of fit" between the images $G : \Omega_0 \to \mathbb{R}$ and $F : \Omega_1 \to \mathbb{R}$, and the second controls the "smoothness" of the mapping $\tilde{u}$. Some other smoothness term could be used, of course. As with the Monge-Kantorovich flows, we can assume $\tilde{u} = u \circ s^{-1}$ where u is an initial mapping from Ω_0 to Ω_1, and s is a family of area-preserving mappings from Ω_0 to itself parameterized by time. Since we are dealing with images, Ω_0 and Ω_1 will both be the same rectangular domain. For this application, one can use the identity map for u, so that $\tilde{u} = s^{-1}$. We also assume that s at time zero is the identity map.

The methodology given above for optimal transport can be used in an analogous manner to derive local and non-local schemes for decreasing the functional (38). One can exploit this type of approach as the basis of scalar field mapping and optical flow algorithms. Finally, other "goodness of fit" terms may be included. In fact, we plan on using mutual information in this context making contact with the work in [72] for registration.

References

[1] L. Ambrosio, "Lecture Notes on Optimal Transport Problems," Lectures given at Euro Summer School, July 2000. Available on http://cvgmt.sns.it/papers/amb00a/.

[2] S. Angenent, S. Haker, A. Tannenbaum, and R. Kikinis, "Laplace-Beltrami operator and brain surface flattening," *IEEE Trans. on Medical Imaging* **18** (1999), pp. 700-711.

[3] S. Angenent, S. Haker, A. Tannenbaum, R. Kikinis, "Laplace-Beltrami operator and brain flattening," *IEEE Trans. Medical Imaging* **18** (1999), pp. 700-711.

[4] S. Angenent, S. Haker, A. Tannenbaum, R. Kikinis, "On area-preserving maps of minimal distortion," in *System Theory: Modeling, Analysis, and Control*, edited by T. Djaferis and I. Schick, Kluwer, Holland, 1999, pages 275-287.

[5] S. Angenent, S. Haker, and A. Tannenbaum, "Minimizing flows for the Monge–Kantorovich problem," *SIAM J. Math. Analysis* **35** (2003) pp. 61-97.

[6] S. Angenent, G. Sapiro, and A. Tannenbaum, "On the affine heat equation for non-convex curves," *Journal of the American Mathematical Society* **11** (1998), pp. 601-634.

[7] J. L. Barron, D. J. Fleet, and S. S. Beauchemin, "Performance of optical flow techniques," *International Journal of Computer Vision* **12** (1994), pp. 43–77.

[8] J.-D. Benamou and Y. Brenier, "A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem," *Numerische Mathematik* **84** (2000), pp. 375-393.

[9] H. Blum, "Biological shape and visual science," *J. Theor. Biology* **38** (1973), pp. 205-287.

[10] Y. Brenier, "Polar factorization and monotone rearrangement of vector-valued functions," *Com. Pure Appl. Math.* **64** (1991), pp. 375-417.

[11] M. Bro-Nielsen and C. Gramkow, "Fast fluid registration of medical images," in *Visualization in Biomedical Imaging* edited by K. Höhne and R. Kikinis, Lecture Notes in Computer Science **1131**, pp. 267-276, Springer-Verlag, New York, 1996.

[12] V. Caselles, F. Catte, T. Coll, and F. Dibos, "A geomteric model for active contours in image processing," *Numerische Mathematik* **66** (1993), pp. 1-31.

[13] V. Caselles, R. Kimmel, and G. Sapiro, "Geodesic snakes," *Int. J. Computer Vision* **22** (1997, pp. 61-79.

[14] J. C. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory*, McMillan, New York, 1991.

[15] G. E. Christensen, R. D. Rabbit, M. Miller, "Deformable templates using large deformation kinetics," *IEEE Trans. on Image Processing* **5** (1996), pp. 1435-1447.

[16] G. E. Christensen, R. D. Rabbit, M. Miller, "A deformable neuroanatomy handbook based on viscous fluid mechanics," *27th Ann. Conf. on Inf. Sciences and Systems*, 1993, pp. 211-216.

[17] M. Cullen and R. Purser, "An extended Lagrangian theory of semigeostrophic frontogenesis," *J. Atmos. Sci.* **41** (1984), pp. 1477-1497.

[18] B. Dacorogna and J. Moser, "On a partial differential equation involving the Jacobian determinant," *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **7** (1990), pp. 1-26.

[19] L. C. Evans, "Partial differential equations and Monge-Kantorovich mass transfer," *Curr. Dev. Math.* (1997), pp. 26-79.

[20] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19**, AMS, Providence, 1998.

[21] M. Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves," *J. Differential Geometry* **23** (1986), pp. 69-96.

[22] M. Grayson, "The heat equation shrinks embedded plane curves to round points," *J. Differential Geometry* **26** (1987), pp. 285–314.

[23] M. Grayson, "Shortening embedded curves," *Annals of Mathematics* **129** (1989), pp. 71–111.

[24] H. W. Guggenheimer, *Differential Geometry*, McGraw-Hill Book Company, New York, 1963.

[25] D. Fry, *Shape Recognition Using Metrics on the Space of Shapes*, Ph.D. Thesis, Harvard University, 1993.

[26] W. Gangbo, "An elementary proof of the polar factorization of vector-valued functions," *Arch. Rational Mechanics Anal.* **128** (1994), pp. 381-399.

[27] W. Gangbo and R. McCann, "The geometry of optimal transportation," *Acta Math.* **177** (1996), pp. 113-161.

[28] W. Gangbo and R. McCann, "Shape recognition via Wasserstein distance," Technical Report, School of Mathematics, Georgia Institute of Technology, 1999.

[29] S. Haker, S. Angenent, A. Tannenbaum, and R. Kikinis, "Nondistorting flattening maps and the 3D visualization of colon CT images," *IEEE Trans. of Medical Imaging* **19** (2000), pp. 665-671.

[30] S. Haker, L. Zhu, A. Tannenbaum, "Optimal mass transport for registration and warping," *Int. Journal Computer Vision* **60** (2004), pp. 225-240.

[31] R. Haralick and L. Shapiro, *Computer and Robot Vision*, Addison-Wesley, New York, 1992.

[32] E. C. Hildreth, "Computations underlying the measurement of visual motion," *Artificial Intelligence*, 23:309–354, 1984.

[33] B. K. P. Horn, *Robot Vision,* MIT Press, Cambridge, Mass., 1986.

[34] B. K. P. Horn and B. G. Schunck, "Determining optical flow," *Artificial Intelligence*, 23:185–203, 1981.

[35] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, and A. Yezzi, "Conformal curvature flows: from phase transitions to active vision," *Archive for Rational Mechanics and Analysis* **134** (1996), pp. 275–301.

[36] T. Kaijser, "Computing the Kantorovich distance for images," *Journal of Mathematical Imaging and Vision* **9** (1998), pp. 173-191.

[37] L. V. Kantorovich, "On a problem of Monge," *Uspekhi Mat. Nauk.* **3** (1948), pp. 225-226.

[38] M. Kass, A. Witkin, and D. Terzopoulos, "Snakes: active contour models," *International Journal of Computer Vision* (1988), pp. 321-331.

[39] B. B. Kimia, A. Tannenbaum, and S. W. Zucker, "Toward a computational theory of shape: An overview", *Lecture Notes in Computer Science* **427**, pp. 402-407, Springer-Verlag, New York, 1991.

[40] B. B. Kimia, A. Tannenbaum, and S. W. Zucker, "Shapes, shocks, and deformations, I," *Int. J. Computer Vision* **15** (1995), pp. 189-224.

[41] B. B. Kimia, A. Tannenbaum, and S. W. Zucker, "On the evolution of curves via a function of curvature, I: the classical case," *J. of Math. Analysis and Applications* **163** (1992), pp. 438-458.

[42] B. B. Kimia, A. Tannenbaum, and S. W. Zucker, "Optimal control methods in computer vision and image processing," in *Geometry Driven Diffusion in Computer Vision,* edited by Bart ter Haar Romeny, Kluwer, 1994.

[43] M. Knott and C. Smith, "On the optimal mapping of distributions," *J. Optim. Theory* **43** (1984), pp. 39-49.

[44] A. Kumar, A. Tannenbaum, and G. Balas, "Optical flow: a curve evolution approach," *IEEE Transactions on Image Processing* **5** (1996), pp. 598–611.

[45] Y. Lauzier, K. Siddiqi, A. Tannenbaum, and S. Zucker, "Area and length minimizing flows for segmentation," *IEEE Trans. Image Processing* **7** (1998), pp. 433-444.

[46] E. Levina and Peter Bickel, "The earth mover's distance is the Mallow's distance: some insights from statistics," *Proceedings IEEE Int. Conf. on Computer Vision*, vol 2., pp. 251-256, July 2001.

[47] R. Malladi, J. Sethian, B. and Vermuri, "Shape modelling with front propagation: a level set approach," *IEEE PAMI* **17** (1995), pp. 158–175.

[48] R. McCann, "Polar factorization of maps on Riemannian manifolds," *Geom. Funct. Anal.*, **11** (2001) pp. 589-608.

[49] R. McCann, "A convexity principle for interacting gases," *Adv. Math.*, **128** (1997), pp. 153-179.

[50] M. Miller, G. Christensen, Y. Amit, U. Grenander, "Mathematical textbook of deformable neuroanatomies," *Proc. National Academy of Science* **90**, pp. 11944-11948.

[51] J. Moser, "On the volume elements on a manifold," *AMS Trans.* **120(2)** (1965), pp. 286-294.

[52] S. J. Osher and J. A. Sethian, "Fronts propagation with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations," *Journal of Computational Physics* **79** (1988), pp. 12-49.

[53] W. Press, S. Teukolsky, W. Vetterling and B. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd Edition, Cambridge University Press, Cambridge U.K., 1992.

[54] S. Rachev and L. Rüschendorf, *Mass Transportation Problems*, Volumes I and II, Probability and Its Applications, Springer, New York, 1998.

[55] A. Rantzer, "A dual to Lyapunov's stability theorem," *Systems and Control Letters*, 2001.

[56] B. ter Haar Romeny (editor), *Geometry–Driven Diffusion in Computer Vision*, Kluwer, Holland, 1994.

[57] Y. Rubner, *Perceptual Metrics for Image Database Navigation*, Ph.D. Thesis, Stanford University, May 1999.

[58] Y. Rubner, C. Tomasi, and J. Guibas, "The earth mover's distance as a metric for image retrieval," Technical Report STAN-CS-TN-98-86, Department of Computer Science, Stanford University, September 1998.

[59] G. Sapiro, *Geometric Partial Differential Equations and Image Analysis* Cambridge University Press, 2001.

[60] G. Sapiro and A. Tannenbaum, "On affine plane curve evolution," *Journal of Functional Analysis* **119** (1994), pp. 79-120.

[61] G. Sapiro and A. Tannenbaum, "Affine invariant scale-space," *International Journal of Computer Vision* **11** (1993), pp. 25–44.

[62] G. Sapiro and A. Tannenbaum, "Invariant curve evolution and image analysis," *Indiana University J. of Mathematics* **42** (1993), pp. 985–1009.

[63] B. G. Schunck, "The motion constraints equation for optical flow," *Proceedings of the Seventh IEEE International Conference on Pattern Recognition*, pp. 20–22, 1984.

[64] J. Shah, "A common framework for curve evolution, segmentation, and anisotropic diffusion," *Proceedings of IEEE CVPR'96*, 1996, pp. 136-142.

[65] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[66] G. Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, Wellesley, Mass., 1986.

[67] A. Tannenbaum, "Three snippets of curve evolution theory in computer vision," *Mathematical and Computer Modelling Journal* **24** (1996), pp. 103–119.

[68] A. Tannenbaum and A. Yezzi, "Visual tracking, active vision, and gradient flows," in *The Confluence of Vision and Control*, edited by G. Hager and D. Kriegman, *Lecture Notes in Control and Information Sciences* **237**, Springer-Verlag, New York, 1998.

[69] D. Terzopoulos and R. Szeliski, "Tracking with Kalman snakes," in *Active Vision* edited by A. Blake and A. Yuille, MIT Press, 1992, page 3-20.

[70] J.-P. Thirion, "Fast non-rigid matching of non-rigid images," *INRIA Technical Report 2547*, Project Epidaure, INRIA, France, 1995.

[71] A. Toga, *Brain Warping*, Academic Press, San Diego, 1999.

[72] W. Wells, P. Viola, and R. Kikinis, "Multi-modal volume registration by maximization of mutual information," *Proceedings of the Second International Symposium on Medical Robotics and Computer Assisted Surgery*, pp.55–62, 1995.

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13. "Stochastic approximations of curve shortening flows" (with G. Ben Arous and O. Zeitouni), *Journal of Differential Equations* **195** (2003), pp. 119-142.

14. "Minimizing flows for the Monge-Kantorovich problem" (with S. Angenent and S. Haker), *SIAM J. Math. Analysis* **35** (2003) pp. 61-97.

15. "Automated identification of RNA conformational motifs: theory and application to the HM LSU 23S rRNA" (with with E. Tannenbaum, E. Herschkowitz, S. Howerton, and L. Williams), *Nucleic Acids Research* **31** (2003), pp. 1-9.

16. "Mathematical methods in medical imaging," (with S. Angenent), invited paper for *Bulletin of the AMS.*

17. "Visual search automation for unmanned aerial vehicles," (with E. Johnson, A. Proctor, and Jincheol Ha), to appear in *IEEE Trans. on Aerospace and Electronic Systems.*

18. "Flattening maps for the visualization of multi-branched vessels" (with S. Haker and L. Zhu), to appear in *IEEE Trans. Medical Imaging.*

19. "Statistically based flow for image segmentation" (with E. Pichon and R. Kikinis), *Medical Image Analysis* **8** (2004), pp. 267-274.

20. "The 3D structure of real polymer foams" (with M. Montminy and C. Macosko), *Journal of Colloidal and Interfacial Science* **280** (2004), pp. 202-211.

21. "Area-based medial axis of planar curves," (with S. Betelu, M. Niethammer, and G. Sapiro), *Int. Journal Computer Vision* **60** (2004), 203-224.

22. "Flux driven automatic centerline extraction" (with S. Bouix and K. Siddiqi), to appear in *Medical Image Analysis*, 2004.

23. "Statistical analysis of RNA backbone" (with E. Hershkowitz, G. Sapiro, and L. Williams), submitted for publication to *IEEE Trans. Computational Biology and Bioinformatics.*

24. "Dynamic geodesic active contours" (with S. Angenent and M. Niethammer), submitted for publication to *IEEE Trans. Automatic control.*

25. "Despeckling of medical ultrasound images" (with O. Michailovich), submitted for publication to *IEEE Trans. Medical Imaging.*

26. "Roles of site-bound magnesium ions in folding of globular RNAs" (with E. Tannenbaum, E. Herschkowitz, S. Howerton, G. Perng, and L. Williams), submitted for publication in *RNA.*

27. "Dynamic denoising of tracking sequences of SAR data" (with O. Michailovich), submitted for publication to *IEEE Trans. Image Processing.*

28. "Fast approximation of smooth Functions from samples of partial derivatives with applications to phase unwrapping" (with O. Michailovich), submitted for publication to *SIAM J. Numerical Analysis.*

29. "A method for prediction and estimation of large-amplitude optical flows - an extended Kalman filtering approach" (with O. Michailovich), submitted for publication to *IEEE Trans. Image Processing.*

30. "On the evolution of closed curves by means of vector distance functions" (with M. Niethammer and P. Vela), submitted for publication to *Int. Journal Computer Vision.*

31. "Detecting simple points in higher dimensions" (with M. Niethammer, W. Kalies, and K. Mischaikow), submitted for publication in *Pattern Recognition.*

32. "Development and testing of highly autonomous unmanned aerial vehicles" (with J. Ha, E. Johnson, and A. Proctor), submitted for publication in *AIAA JACIC.*

33. "Automatic recognition of electronic components" (with G. Pryor and A. Goldstein), *Circuits Assembly* **15** (2004), pp. 36-40.

34. "Advanced nonlinear registration algorithms for image fusion" (with S. Warfield *et al*), in *Brain Mapping: The Methods, Second Edition* edited by Arthur Toga and John Mazziotta, Academic Press, pages 661-690, 2003.

35. "Mean curvature flows, edge detection, and medical image segmentation" (with S. Angenent, S. Haker, and A. Yezzi), in *Computational Methods in Biophysics, Biomaterials, Biotechnology and Medical Systems* edited by C. Leondes, pages 253-269, Kluwer, 2003.

36. "New approach to Monge-Kantorovich with applications to computer vision and image processing" (with S. Haker), IMA Series on Applied Mathematics, volume 133, Springer-Verlag, New York, 2003.

37. "Maximal entropy reconstruction of back projection images" (with T. Georgiou and P. Olver), IMA Series on Applied Mathematics, volume 133, Springer-Verlag, New York 2003.

38. "Segmentation of diffusion tensor images" (with Eric Pichon and Guillermo Sapiro), in *Directions in Mathematical Systems Theory and Optimization* edited by Anders Rantzer and Chris Byrnes, pages 240-249, Springer, New York, 2002.

39. "Optimal image interpolation and optical flow" (with S. Haker), in *Multidisciplinary Research in Control, Lecture Notes in Control and Inform. Sci* **289**, pages 133-143, 2002.

40. "Stochastic crystalline flows" (with G. Ben-Arous and Ofer Zeitouni), in *Mathematical Systems Theory in Biology, Communications, Computation, and Finance* edited by J. Rosenthal and D. Gilliam, IMA Volumes in Mathematics and Its Applications, volume 134, pages 41-63, Springer, New York, 2003.

41. "On a stochastic model of geometric snakes" (with D. Nain, G. Unal, A. Yezzi, and O Zeitouni), to appear in *Mathematical Methods in Computer Vision: A Handbook*, edited by O. Faugeras and N. Paragios, Springer-Verlag, 2005.

42. "Robust control and tracking", *Proceedings of IEEE CDC'00.*

43. "Affine invariant symmetry sets" (with S. Betelu and G. Sapiro), *Proceedings of ECCV'00*, Dublin, Ireland, June 2000.

44. "Nondistorting maps for virtual colonoscopy" (with S. Angenent, S. Haker, and R. Kikinis), *Proceedings of SPIE*, San Diego, February 2000.

45. "New approach for visualization of 3D colon imagery" (with S. Angenent, S. Haker, and R. Kikinis), *MICCAI'00*, October 2000.

46. "Length-based attacks for certain group based encryption rewriting systems" (with J. Hughes), *SECI'02 Conference, Securité de la Communication sur Internet*, September 2002.

47. "New algorithms for 3D analysis of open-celled foams," (with M. Montminy and C. Macosko), *Proceedings of FOAM 2000*, New Jersey.

48. "High resolution sensing and anisotropic segmentation for SAR imagery" (with T. Georgiou), *Proceedings of IEEE CDC'00.*

49. "Affine invariant erosion for 3D shapes" (with S. Betelu and G. Sapiro), *ICCV'01*, 2001.

50. "Missile tracking using knowledge-based adaptive thresholding" (with S. Haker, G. Sapiro, and D. Washburn), *ICIP'01*, 2001.

51. "Cubical homology and the topological classification of 2D and 3D imagery" (with M. Allili and K. Mischaikow), *ICIP'01*, 2001.

52. "Optimal transport and image warping" (with S. Haker), *IEEE Conference on Variational and Level Set Methods in Computer Vision*, Vancouver, 20001.

53. "Mass-preserving mappings and surface registration" (with S. Haker and R. Kikinis), *MICCAI'01*, October 2001.

54. "Minimal transport for nonlinear control" (with S. Haker), *CDC'01*, December 2001.

55. "L^1 based optical flow for cardiac wall motion tracking" (with A. Kumar, S. Haker, A. Stillman, C. Curry, D. Giddens, and A. Yezzi), *Proceedings of SPIE*, San Diego, February 2001.

56. "Visual tracking and object recognition" (with A. Yezzi and A. Goldstein), *Proceedings of NICOLS'01*, St. Petersburg, Russia, July, 2001.

57. "Conformal flattening maps for the visualization of vessels" (with S. Haker and L. Zhu), *Proceedings of SPIE*, San Diego, 2002.

58. "Cubical topological analysis of blood vessels" (with M. Niethammer and A. Stein), *Proceedings of ICIP*, 2002.

59. "Angle-reserving mappings and multiply branched vessels" (with L. Zhu and S. Haker), *Proceedings of ICIP*, 2002.

60. "4D active surfaces for MR cardiac analysis" (with A. Yezzi), *Proceedings of MICCAI'02.*

61. "Flux driven fly-throughs," *CVPR*, 2003.

62. "A Stokes flow based boundary integral formulation for measuring crosssections of two-dimensional tubular structures," (with M. Niethammer and E. Pichon), *ICIP*, 2003.

63. "Algorithms for stochastic approximations of curvature flows" (with G. Ben-Arous, N. Shimkin, G. Unal, and O. Zeitouni), *ICIP*, 2003.

64. "Dynamic level sets for visual tracking" (with M. Niethammer), *IEEE Conference on Decision and Control*, 2003.

65. "Statistically based surface evolution method for medical image segmentation: presentation and validation" (with E. Pichon and R. Kikinis), *MICCAI*, 2003. (Based student presentation award.)

66. "Active contours and optical flow for automatic tracking of flying vehicles" (with J. Ha, C. Alvino, G. Pryor, E. Johnson), *American Control Conference*, 2004.

67. "Image interpolation based on optimal mass preserving maps" (with L. Zhu), *Proceeedings of ISBI*, 2004.

68. "Flux driven fly-throughs" (with S. Bouix and K. Siddiqi) *Proceedings of CVPR*, 2003.

69. "Dynamic geodesic snakes" (with M. Niethammer), in *Proceedings of CVPR*, 2004.

70. "Knowledge-based 3D segmentation and reconstruction of left coronary arteries using CT images" (with D. Giddens and Y. Yang), *EMBS04*, 2004.

71. "Flow patterns and wall shear stress distributions at atherosclerotic-prone sites in a human left coronary artery-an exploration using combined methods of CT and computational fluid dynamics (with S. Jin, Y. Yang, J. Oshinski, A. Tannenbaum, J. Gruden, and D. Giddens), *EMBS04*, 2004.

72. "Image morphing based on mutual information and optimal mass transport" (with L. Zhu), to appear in *Proceedings of ICIP*, 2004.

73. "Automatic tracking of flying vehicles using geodesic snakes and Kalman filtering" (with A. Betser and P. Vela), *IEEE CDC*, 2004.

74. "Flying in formation using a pursuit guidance algorithm" (with A. Betser, G. Pryor, and P. Vela), submitted for publication to *American Control Conference*, 2005.

75. "Tracking moving and deforming shapes using a particle filter" (with Y. Rathi, N. Vaswani, A. Yezzi), submitted for publication to *CVPR*, 2005.

76. "Affine surface evolution for 3D segmentation" (with Y. Rathi, P. Olver, G. Sapiro), submitted for publication to *ICIP*, 2005.

77. "Pattern detection and image segmentation with anisotropic conformal factors" (with E. Pichon), submitted for publication to *ICIP*, 2005.

Books Written Under AFOSR Support

1. *Invariance and Systems Theory: Algebraic and Geometric Aspects, Lecture Notes in Mathematics* **845**, Springer-Verlag, New York, 1981.
2. *Feedback Control Theory* (with John Doyle and Bruce Francis), MacMillan Company, New York, 1991. (This book has been translated into Chinese and Japanese.)
3. *Robust Control of Distributed Parameter Systems* (with Ciprian Foias and Hitay Özbay), *Lecture Notes in Control and Information Sciences* **209**, Springer-Verlag, New York, 1995.
4. *Feedback Control, Uncertainty, and Complexity*, edited by Bruce Francis and Allen Tannenbaum, *Lecture Notes in Control and Information Sciences* **202**, Springer-Verlag, New York, 1995.
5. *Deformation Theory*, lectures by Michael Artin, notes by C. S. Seshadri and A. Tannenbaum, *Tata Institute Lecture Notes,* Bombay, India, 1976.
6. *Curvature Flows, Visual Tracking, and Computational Vision,* to be published by *SIAM.*
7. *Mathematical Methods in Computer Vision*, edited by Peter Olver and Allen Tannenbaum, *IMA Series on Applied Mathematics*, volume 133, Springer-Verlag, 2004.

Patents Based on AFOSR Projects

1. "Conformal Geometry and Texture Mappings," (co-inventors Sigurd Angenent, Steven Haker, Allen Tannenbaum, and Ron Kikinis), U.S. Patent Number 6,697,538, issued February 24, 2004.
2. "4D Kappa5 Gaussian Noise Reduction," (co-inventors Harvey Cline and Allen Tannenbaum), U.S. Patent Number 6,204,853 B1, issued March 20, 2001.
3. "Curvature Based System for the Segmentation and Analysis of Cardiac Magnetic Resonance Imagery," (co-inventors Allen Tannenbaum and Anthony Yezzi), U.S. Patent Number 6,535,623 B1, issued March 18, 2003.

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1. Allen Tannenbaum (faculty)
2. Marc Niethammer (Ph.D. granted December 2004)
3. Eric Pichon (Ph.D. student; expected date of graduation: September 2005)
4. Andrew Stein (M.S. granted December 2002)